Chapter 5.3: The Definite Integral

Limits of Riemann Sums using $\int$

Computing the area under $f(x)$ for $x \in [a, b]$: Pick $a = a_0 < \cdots < a_n = b$ and $a_{k-1} \le x_i \le a_k$ and $\Delta_k = a_k - a_{k-1}$.

$$\text{area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta_k$$

Notation using the *definite integral*

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta_k$$

to b

in variable x.

Integral $\int_a^b f(x)\, dx$

from a of function $f(x)$

Few things to notice

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta_k$$

If the limit exists, f is called *integrable*.

All continuous functions and functions with finitely many jumps are integrable.

Line has an orientation $a < b$. Flipping bounds flips sign.

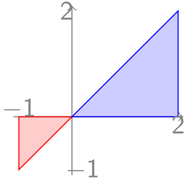
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Recall: Area if $f(x) < 0$, the area between $f(x)$ and the axis is negative.

Easy examples

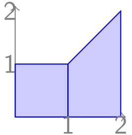
Evaluate the following integrals

- $\int_{-1}^{2} x\, dx = \frac{3}{2}$
  The blue area is $(1/2)(2)(2) = 2$ and the red area is $(1/2)(1)(1) = 1/2$. Adding the blue to *negative* the red yields $\frac{3}{2}$.

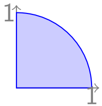


- $\int_{0}^{2} f(x)\, dx$, where $f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ x & 1 < x \le 2 \end{cases}$

$$\int_0^2 f(x)\, dx = 1 + \frac{3}{2} = \frac{5}{2}$$



- $\int_{0}^{1} \sqrt{1-x^2}\, dx = \frac{\pi}{4}$
  Notice the area is a quarter of a circle.



Properties of integration

- $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$
- $\int_a^a f(x)\, dx = -\int_a^a f(x)\, dx = 0$
- $\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$
- $\int_a^b c \cdot f(x)\, dx = c \cdot \int_a^b f(x)\, dx$
- $\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$
- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq M$.

More examples

Example: Find $\int_1^5 f(x)\,dx$ given

$$\int_1^3 f(x)\,dx =$$

$$\int_2^3 f(x)\,dx =$$

$$\int_2^5 f(x)\,dx =$$

See the graphical explanation.

Example: Given that

$$\int_4^7 f(x)\,dx =$$

$$\int_4^7 g(x)\,dx =$$

$$\int_4^7 \left(3 \cdot f(x) + 2 \cdot g(x)\right)\,dx =$$

Even more examples

$$\int_{-5}^{5} \frac{t^3}{t^4+t^2+1}\,dt = 0$$

Notice the function is odd. That means $f(x) = -f(-x)$.

The *average value* of $f(x)$ on $[a, b]$ is

$$\boxed{\frac{1}{b-a}\int_a^b f(x)\,dx}$$

Find the average value of $f(x) = \sqrt{1-x^2}$ on $[-1, 1]$.

Notice that $f(x)$ will be a halfcircle. Hence

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)\,dx \\
&= \frac{1}{1-(-1)}\int_{-1}^{1} f(x)\,dx \\
&= \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}
\end{aligned}$$

Mean Value Theorem Again

Let f be continuous on $[a, b]$. Then there exists a c in $[a, b]$ such that

$$f(c) = \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\text{Average value of } f(x) \text{ on } [a,b]}$$

Idea: Use the Intermediate Value Theorem.
Let m be the minimum of $f(x)$ on $[a, b]$.
Let M be the maximum of $f(x)$ on $[a, b]$.

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a),$$

$$m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M$$

and so the Intermediate Value Theorem yields the existence of the desired c.